# CH 29 – COMPLETING THE SQUARE

## ❑ INTRODUCTION

Consider the quadratic equation $x^2 + 5x + 1 = 0$. It doesn't factor using integers (confirm this), so how do we solve this equation? You may recall some "formula" from an earlier Algebra course, but in this chapter we employ a neat trick called "Completing the Square," a technique that not only solves quadratic equations, but is also used in a future chapter to find the center and radius of a circle.

## ❑ THE "MAGIC NUMBER"

Consider the trinomial $x^2 + 10x + 25$. We've learned that its factorization is

$$x^2 + 10x + 25 = (x + 5)^2$$, which is the square of a binomial.

Let's look carefully at the numbers in this equality. Notice that the 5 is <u>half</u> of the 10, and that the 25 is the <u>square</u> of the 5.

Let's do one more example. Consider the factorization

$$n^2 - 14n + 49 = (n - 7)^2$$, which is the square of a binomial.

We note that the −7 is <u>half</u> of the −14, and that the 49 is the <u>square</u> of the −7.

So now imagine that I give you

$$x^2 + 6x,$$

and I ask you to add a third term to this binomial so that the resulting trinomial will factor into the square of a binomial. In other words,

$$x^2 + 6x + ??? = (x + ?)(x + ?)$$

Each single "?" must be 3, since 3 is half of 6. Also, the "???" must therefore be 9, since 9 is the square of the 3. In other words,

$$x^2 + 6x + \underline{\mathbf{9}} = (x + \underline{\mathbf{3}})(x + \underline{\mathbf{3}})$$

We shall call 9 the "magic number." It can be calculated for this problem using the following two-step rule:

1) Calculate <u>half</u> of 6, which is **3**.
2) <u>Square</u> the 3, giving **9**, the "magic number."

**Note:** When we convert $x^2 + 6x$ to $x^2 + 6x + 9$ by adding the "magic number" 9, we are <u>not</u> saying that they're equal, but there will always be a way of adding the magic number without violating any of the laws of algebra.

To become proficient in completing the square, we must be really good at finding the *magic number*. The following chart gives more examples of how this is done. We'll let $b$ represent the number in front of the variable (the coefficient of the linear term), as in $ax^2 + bx + c$.

| Original Quadratic | Value of *b* | <u>Half</u> of *b* | Half of *b* <u>Squared</u> = The Magic Number | New Quadratic | New Quadratic in Factored Form |
|---|---|---|---|---|---|
| $x^2 + 22x$ | 22 | 11 | **121** | $x^2 + 22x + \mathbf{121}$ | $(x + 11)^2$ |
| $A^2 - 14A$ | $-14$ | $-7$ | **49** | $A^2 - 14A + \mathbf{49}$ | $(A - 7)^2$ |
| $n^2 + 9n$ | 9 | $\frac{9}{2}$ | $\mathbf{\frac{81}{4}}$ | $x^2 + 9x + \mathbf{\frac{81}{4}}$ | $\left(x + \frac{9}{2}\right)^2$ |
| $y^2 - \frac{2}{5}y$ | $-\frac{2}{5}$ | $-\frac{1}{5}$ | $\mathbf{\frac{1}{25}}$ | $y^2 - \frac{2}{5}y + \mathbf{\frac{1}{25}}$ | $\left(y - \frac{1}{5}\right)^2$ |
| $3u^2 - 7u$ | This problem can't be done yet; the leading coefficient is <u>not</u> 1. | | | | |

# Homework

1. Find the **magic number** for each quadratic binomial:

a. $a^2 + 8a$ b. $x^2 - 12x$ c. $y^2 + 28y$

d. $b^2 + 3b$ e. $c^2 - 7c$ f. $t^2 + 11t$

g. $x^2 + x$ h. $y^2 - y$ i. $z^2 + 2z$

j. $g^2 + \frac{2}{3}g$ k. $a^2 - \frac{5}{7}a$ l. $x^2 - \frac{7}{13}x$

## ❑ *THE FIVE STEPS IN COMPLETING THE SQUARE*

We've reviewed or learned three skills so far in this chapter and the previous chapter:

- Factoring perfect square trinomials
- Solving quadratic equations using the Square Root Theorem
- Finding the "magic number"

Putting all these skills together will allow us to solve all kinds of quadratic equations, including the type where we could have factored, and the type mentioned in the Introduction where normal factoring is impossible. The following is the general scheme for solving any quadratic equation by "completing the square."

We start by assuming that the quadratic equation is in standard form:

$$ax^2 + bx + c = 0 \qquad \text{(where, of course, } a \neq 0\text{)}$$

1. Make sure that the leading coefficient (the *a*) is 1. Divide each side of the equation by *a*, if necessary.
2. Move the constant to the other side of the equation.
3. Compute the "magic number" and add it to both sides of the equation. This step "*completes the square*."
4. Factor the left side, and then simplify the right side.
5. Solve the resulting equation by taking square roots, remembering that every positive number has two square roots (the Square Root Theorem).

Note: Steps 1 and 2 can be done in either order.

## ❑ EXAMPLES OF COMPLETING THE SQUARE

EXAMPLE 1: **Solve by Completing the Square:**

$$\boldsymbol{x^2 + 8x - 20 = 0}$$

Solution: Even though this quadratic equation is factorable, we'll solve it by completing the square, a technique which will also work on quadratics which aren't factorable, as well as future problems involving the center and radius of a circle.

Step 1:

We must make sure that the leading coefficient (the $a$) is 1. It already is 1 ($x^2 = 1x^2$), so step 1 is done, and the equation remains the same:

$$x^2 + 8x - 20 = 0$$

Step 2:

Move the constant (the –20) to the right side of the equation by adding 20 to each side of the equation:

$$x^2 + 8x = 20$$

Step 3:

Now we calculate the **magic number**:

a. Calculate half of 8: $\frac{1}{2}(8) = 4$

b. Square the 4; the magic number is **16**.

Add the magic number to each side of the equation:

$$x^2 + 8x + \boxed{16} = 20 + \boxed{16}$$

Step 4:

Factor the left side, and simplify the right side:

$$(x + 4)^2 = 36$$

Step 5:

Solve by taking square roots:

$$x + 4 = \pm\sqrt{36} \qquad \text{(36 has two square roots)}$$

$$x + 4 = \pm 6 \qquad \text{(simplify the radical)}$$

$$x = -4 \pm 6 \qquad \text{(subtract 4 from both sides)}$$

The plus sign $\Rightarrow x = -4 + 6 = 2$

The minus sign $\Rightarrow x = -4 - 6 = -10$

We have now solved our first equation by completing the square, and its solutions are

$$\boxed{x = 2, -10}$$

EXAMPLE 2: **Solve by Completing the Square:**

$$3x^2 - 5x + 1 = 0$$

Solution: This quadratic will not factor, so we have no choice but to use completing the square, since these are the only two techniques we have learned in this class.

Step 1:

The first requirement for completing the square is to have a leading coefficient of 1. Since the leading coefficient in this problem is 3, we will divide both sides of the equation by 3:

$$\frac{3x^2 - 5x + 1}{3} = \frac{0}{3}$$

or, $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$ (divide all terms by 3)

Step 2:

Moving the constant to the right side produces the equation

$$x^2 - \frac{5}{3}x = -\frac{1}{3} \qquad \text{(subtract } \tfrac{1}{3} \text{ from both sides)}$$

Step 3:

Now for the "magic number," the number that will "complete the square." We calculate half of $-\frac{5}{3}$, square that result, and we'll have the "magic number" that will be added to each side of the equation.

Magic Number Calculation: $\frac{1}{2}\left(-\frac{5}{3}\right) = -\frac{5}{6}$, and then $\left(-\frac{5}{6}\right)^2 = \mathbf{\frac{25}{36}}$

Add the magic number to each side of the equation:

$$x^2 - \frac{5}{3}x + \boxed{\frac{25}{36}} = -\frac{1}{3} + \boxed{\frac{25}{36}}$$

Step 4:

Factoring the left side and adding on the right gives:

$$\left(x-\frac{5}{6}\right)^2 = \frac{13}{36} \qquad \left[-\frac{1}{3}+\frac{25}{36} = -\frac{12}{36}+\frac{25}{36} = \frac{13}{36}\right]$$

Step 5:

Now we take the square root of each side of the equation, remembering that the right side has two square roots:

$$x-\frac{5}{6} = \pm\sqrt{\frac{13}{36}}$$

Let's isolate the $x$ and simplify the radical at the same time:

$$x = \frac{5}{6}\pm\frac{\sqrt{13}}{6}$$

Combining the fractions into a single fraction (the LCD is 6) produces the final answer:

$$\boxed{x = \frac{5\pm\sqrt{13}}{6}}$$

Our quadratic has two solutions.

# Homework

2. Solve each quadratic equation by Completing the Square:

a. $y^2 - 6y + 5 = 0$

b. $x^2 + 13x + 30 = 0$

c. $z^2 + 5z - 14 = 0$

d. $2n^2 - n - 1 = 0$

e. $12t^2 - 5t - 3 = 0$

f. $10a^2 + 7a + 1 = 0$

g. $x^2 + 25 = 10x$

h. $4u^2 + 20u + 25 = 0$

i. $w^2 = -w - 5$

j. $2h^2 + 1 = h$

3. Solve each quadratic equation by Completing the Square:

a. $x^2 + 3x + 1 = 0$

b. $y^2 - 4y + 2 = 0$

c. $2a^2 + 6a + 3 = 0$

d. $n^2 + 8n - 2 = 0$

e. $3u^2 - 4u - 2 = 0$

f. $t^2 + 10t + 3 = 0$

g. $5w^2 + w + 1 = 0$

h. $2x^2 = -5x - 1$

i. $g^2 = 3g - 5$

j. $3m^2 = 1 - 4m$

4. Solve each equation by Completing the Square:

a. $q^2 + 8q + 15 = 0$

b. $2x^2 - 7x + 1 = 0$

c. $3n^2 + n - 5 = 0$

d. $4a^2 - 5a = 3$

e. $5a^2 = -2 - 7a$

f. $z^2 + 1 = 2z$

# Solutions

**1.** a. $\frac{1}{2}\cdot 8 = 4$, and $4^2 = \mathbf{16}$

b. $\frac{1}{2}\cdot -12 = -6$, and $(-6)^2 = \mathbf{36}$

c. $\frac{1}{2}\cdot 28 = 14$, and $14^2 = \mathbf{196}$

d. $\frac{1}{2}\cdot 3 = \frac{3}{2}$, and $\left(\frac{3}{2}\right)^2 = \mathbf{\frac{9}{4}}$

e. $\frac{1}{2}\cdot -7 = -\frac{7}{2}$, and $\left(-\frac{7}{2}\right)^2 = \mathbf{\frac{49}{4}}$

f. $\frac{1}{2}\cdot 11 = \frac{11}{2}$, and $\left(\frac{11}{2}\right)^2 = \mathbf{\frac{121}{4}}$

g $\frac{1}{2}\cdot 1 = \frac{1}{2}$, and $\left(\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

h. $\frac{1}{2}\cdot -1 = -\frac{1}{2}$, and $\left(-\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

i. $\frac{1}{2}\cdot 2 = 1$, and $1^2 = \mathbf{1}$

j. $\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$, and $\left(\frac{1}{3}\right)^2 = \mathbf{\frac{1}{9}}$

k. $\frac{1}{2} \cdot -\frac{5}{7} = -\frac{5}{14}$, and $\left(-\frac{5}{14}\right)^2 = \mathbf{\frac{25}{196}}$

l. $\frac{1}{2} \cdot -\frac{7}{13} = -\frac{7}{26}$, and $\left(-\frac{7}{26}\right)^2 = \mathbf{\frac{49}{676}}$

**2.** a. $y = 1, 5$ b. $x = -3, -10$ c. $z = 2, -7$
d. $n = 1, -\frac{1}{2}$ e. $t = \frac{3}{4}, -\frac{1}{3}$ f. $a = -\frac{1}{2}, -\frac{1}{5}$
g. $x = 5$ h. $u = -\frac{5}{2}$ i. No solution in $\mathbb{R}$
j. No solution in $\mathbb{R}$

**3.** a. $x = \frac{-3 \pm \sqrt{5}}{2}$ b. $y = 2 \pm \sqrt{2}$ c. $a = \frac{-3 \pm \sqrt{3}}{2}$
d. $n = -4 \pm 3\sqrt{2}$ e. $u = \frac{2 \pm \sqrt{10}}{3}$ f. $t = -5 \pm \sqrt{22}$
g. No solution in $\mathbb{R}$ h. $x = \frac{-5 \pm \sqrt{17}}{4}$ i. No solution in $\mathbb{R}$
j. $m = \frac{-2 \pm \sqrt{7}}{3}$

**4.** a. $q = -3, -5$ b. $x = \frac{7 \pm \sqrt{41}}{4}$ c. $n = \frac{-1 \pm \sqrt{61}}{6}$
d. $a = \frac{5 \pm \sqrt{73}}{8}$ e. $a = -1, -\frac{2}{5}$ f. $z = 1$

*"Learning is a treasure that will follow its owner everywhere."*

Chinese Proverb